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# Evaluation on the equivalence between the backward/forward iterative sweep and the triangular-based power flow methods in electrical distribution networks

# Sobre la equivalencia entre el barrido iterativo hacia atrás /adelante y los métodos de flujo de potencia basados en triángulos en redes de distribución eléctrica

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**Abstract**

This paper demonstrates that graph-based power flow methods for strictly radial distribution networks (one based on the upper-triangular matrix and the other on the incidence matrix or classical backward/forward power flow method) are equivalent, implying that both recursive power flow formulas are the same. A small distribution network composed of seven nodes and six distribution lines is considered to demonstrate this equivalence. The main contribution of this research lies in the fact that it obtains a matrix relation between the upper triangular matrix and the demand-to-demand branch-to-node incidence matrix. Numerical comparisons in single-phase distribution networks comprising 33, 34, 69, and 85 nodes and three-phase asymmetric networks comprising 8, 25, and 37 nodes confirm the theoretical results.

**Keywords:** Upper-triangular matrix; branch-to-node incidence matrix; power flow methods; equivalent formulation.

**Resumen**

Este breve documento demuestra que los métodos de flujo de potencia basados en grafos para redes de distribución estrictamente radiales (uno basado en la matriz triangular superior y otro en la matriz de incidencia o método clásico de flujo de potencia hacia atrás/adelante) son equivalentes, lo que implica que ambas fórmulas de flujo de potencia recursivas son iguales. Se considera una pequeña red de distribución compuesta por siete nodos y seis líneas de distribución para demostrar esta equivalencia. La principal contribución de esta investigación radica en el hecho de que obtiene una relación matricial entre la matriz triangular superior y la matriz de incidencia de rama a nodo de demanda a demanda. Comparaciones numéricas en redes de distribución monofásicas que comprenden 33, 34, 69 y 85 nodos y redes asimétricas trifásicas que comprenden 8, 25 y 37 nodos confirman los resultados teóricos.

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**Palabras clave:** Matriz triangular superior; matriz de incidencia de rama a nodo; métodos de flujo de potencia; formulation equivalente.

## 1. Introduction

Power flow analysis in electrical networks has been the most studied problem in electrical engineering for more than six decades [1]. It is a steady-state study whose main interest is to determine the voltage magnitudes and angles in all the demand nodes of the network by using recursive solution approaches (i.e., iterative methods) [2]. When these voltages are determined, the electrical behavior of the network is completely determined, and it is possible to know the grid efficiency, voltage regulation, line chargeability, and power factor in generation sources, among other calculations.

The most widely known power flow technique for power systems is the Newton-Raphson power flow method, which exhibits excellent behavior in strong meshed distribution networks with multiple generation sources [3]. However, in electrical distribution networks, due to their tree-based configuration (radial topology) and the presence of only one substation bus (slack), most power flow methodologies existing in the literature are founded upon graph theory. These solution techniques include the backward/forward sweep power flow method [4], the successive approximations power flow method [5], and the upper triangular power flow formulation [6], among others. The main characteristic of these graph-based approaches is that they are derivative-free, with linear convergence and faster processing times [7].

The main goal of this paper is not to propose a new power flow formulation for radial distribution networks but to demonstrate that the classical backward/forward power flow (BFPF) method is mathematically equivalent to the upper-triangular power flow (UTPF) approach. To this effect, the equivalence between the demand-todemand branch-to-node incidence matrix and the uppertriangular matrix is proved. The theoretical analysis presented in this research confirms that the recursive formula used in the BFPF method and the one used in the UTPF approach are the same.

The remainder of this document is organized as follows: Section 2 presents the general derivation of the UTPF formula, Section 3 describes the general formulation of the BFPF approach, Section 4 presents the theoretical demonstration that both power flow formulations are entirely equivalent for strictly radial distribution grids; Section 5 presents the demonstration of convergence of the UTPF formula using the Banach fixed-point theorem; and Section 6 presents the numerical confirmation of the theoretical achievements by offering simulations in the singleand three-phase IEEE systems composed of 8, 25, 33, 34, 37, 69, and 8 nodes, respectively. Finally, Section 6 concludes that both power flow approaches are mathematically equivalent.

## 2. Upper-Triangular Power Flow Method

The upper-triangular power flow method (UTPF) is a graphbased power flow method originally proposed by [8] and extended to three-phase distribution networks by [9] and to meshed configurations by [10] with detailed dispersed generation models. The main idea of this technique is to exploit the graph-based structure of radial distribution networks, which allows representing the branch and nodal currents and voltages by using an upper-triangular matrix [7].

Lemma 1. *The UTPF method is a graph-based power flow approach that allows determining all the voltage values at the demand nodes of a strictly radial distribution grid with only one substation bus by using the following recursive power flow formula*:

$$\mathbb{V}_d^{t+1} = \mathbf{1}_d \mathbb{V}_s - T^\top \mathbf{Z}_{bb} \mathbf{T} diag^{-1}\left(\mathbb{V}_d^{t,\star}\right) \mathbb{s}_d^\star, \qquad (1)$$

where $\mathbb{V}_d$ is a vector with dimension d that contains all the unknown demand voltages; $1_d$ is a vector with dimensión d that is filled with ones; $\mathbb{V}_s$ is the voltage output at the terminals of the substation; $\mathbf{T}$ is a square matrix with an upper-triangular structure containing values of zero and one; $\mathbf{Z}_{bb}$ is a square diagonal matrix with $b \times b$ dimension that contains all the primitive impedance values of all the distribution branches; and $\mathbb{s}_d^\star$ is the vector with dimension d that contains all the complex values of the demanded loads. Note that diag ($X$) is a function that turns a vector into a diagonal matrix with appropriate dimensions and $X\star$ is the conjugate value for the vector $X$. In addition, t corresponds to the iterative counter.

Proof. To demonstrate the recursive power flow formula for the UTPF method presented in Equation (1), let us start with a small numerical example while considering the radial distribution network in Figure 1.

In Figure 1, it can be observed that the current flow through the branch $l$ can be represented as a function of the demanded currents at nodes downstream of the branch $l$, *i.e.*, $I_k$, as presented below.

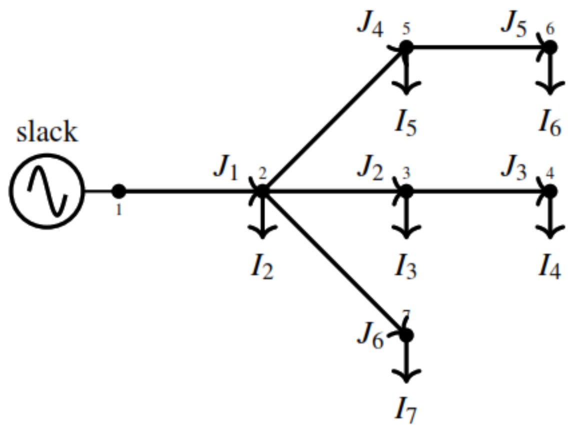


Figure 1. Single-line diagram equivalent of a three-phase distribution system composed of 7 buses.

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \\ J_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix}$$

This matricial relation between branch and nodal currents can be represented in a compact manner as defined in (2):

$$\mathbb{J}_b = T\mathbb{I}_d, \tag{2}$$

where $\mathbb{J}_b \in C^{bx1}$ is the vector with dimension $b$ that contains all the branch currents; $\mathbb{I}_d \in C^{bx1}$ is a vector with dimension d that contains all the currents at the demand nodes, excluding the net injected current in the slack source. Note that $\mathbf{T} \in \mathcal{R}^{bxb}$ represents the upper triangular matrix.

Now, the voltage value at node $k$, $\mathbb{V}_k$, the voltage output in the slack source, $\mathbb{V}_1$ (node 1 in Figure 1), and the voltage drop in branch $l$, $E_l$, can be related as follows (note that this set of equations is obtained after applying Kircchoff's second law for the closed-loop trajectory from the substation to each demand node):

$$\begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} V_1 - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix}$$

This matricial relation between nodal and Branch voltages can be represented in a compact manner as defined in (3):

$$\mathbb{V}_d = \mathbf{1}_d\mathbb{V}_s - T^{\top}\mathbb{E}_b, \tag{3}$$

where $\mathbb{E}_b \in C^{b\times 1}$ is a vector with dimension b that contains the voltage drops in all the distribution lines, $\mathbb{V} \in C^{b\times 1}$.

By applying Ohm's law to each branch, it is possible to find the relation between the voltage drops in the branches and the current flow, as presented below:

$$\mathbb{E}_b = \boldsymbol{Z}_{bb}, \mathbb{J}_b \tag{4}$$

where $\mathbb{Z} \in C^{b\times b}$ is a diagonal matrix with complex elements that corresponds to the impedances of the branches, *i.e.*, **Z** = diag ([Z1, Z2, · · ·, Z6]).

Now, in order to relate the voltage variables in the demand nodes and the current demands, i.e., $\mathbb{V}_d$ and $\mathbb{I}_d$, Equations (4) and (2) are replaced into (3), which yields the following relation:

$$\mathbb{V}_d = \mathbf{1}_d\mathbb{V}_s - T^{\top}\boldsymbol{Z}_{bb}\mathbb{I}_d. \tag{5}$$

Note that, in the case of single-phase distribution networks, the demanded currents can be expressed as a function of the voltage values and complex power consumptions, as presented in (6).

$$\mathbb{I}_d = \boldsymbol{diag}^{-1}(\mathbb{V}_d^{\star})\mathbb{s}_d^{\star}. \tag{6}$$

Finally, if Equation (6) is replaced into (5), a general recursive nonlinear power flow formula based on the upper triangular matrix is reached, as presented in (7).

$$\mathbb{V}_d = \mathbf{1}_d\mathbb{V}_s - T^{\top}\mathbf{Z}_{bb}\mathbf{T}diag^{-1}(\mathbb{V}_d^{\star})\mathbb{s}_d^{\star}, \tag{7}$$

whose solution requires the addition of an iterative counter $t$ that allows reaching the recursive UTPF formula defined in Equation (1), which completes the demonstration.

## 3. Backward/Forward Power Flow

The backward/forward power flow method (BFPF) is a graph-based power flow method originally proposed by [11] which uses a recursive solution procedure in order to find the voltage values in the demand nodes by using a backward sweep of currents from the end nodes until the current in the line leaving the substation bus is reached. With these currents, a forward sweep is performed to update all the voltage values at the demand nodes. This procedure is repeated until the desired convergence is found. The authors of [4] presented a general reformulation of the BFPF method that uses the branch-to-node incidence matrix, which allows presenting this solution method in compact form.

Lemma 2. *The BFPF method is a graph-based power flow approach that allows determining all the voltaje values at the demand nodes of a radial distribution grid with only one substation bus by using the following recursive power flow formula:*

$$V_d^{t+1} = -[\mathbf{A}_d^{\mathrm{T}}\mathbf{Y}_{\mathrm{bb}}\mathbf{A}_d]^{-1}\big(\mathbf{A}_d^{\mathrm{T}}\mathbf{Y}_{bb}\mathbf{A}_s\mathbb{V}_s + diag^{-1}\big(V_d^{t,\star}\big)S_d^{\star}\big), \qquad (8)$$

where $\mathbb{A}_d$ is the component of the square incidence matrix that relates the demand nodes and the branch currents; $\mathbb{A}_s$ is a rectangular matrix obtained from the branchto-node incidence matrix that relates the slack node with the current in the branches; and $\mathbb{Y}_{bb}$is a square diagonal matrix with $b\times b$ dimension that contains all the primitive admittance values of all the distribution branches.

Proof. To demonstrate that the BFPF method has the recursive formula presented in (8), let us remember the general structure of the branch-to-node incidence matrix [12]:

i. $A_{lk} = 1$ if the current through the line $l$ is leaving the node $k$.
ii. $A_{lk} = -1$ if the current through the line $l$ arrives to the node $k$.
iii. $A_{lk} = 0$ if the the line $l$ is not connected to the node $k$.

Now, considering the small distribution system presented in Figure 1, the branch-to-node incidence matrix for this system is defined below:

$$\mathbb{A} = [\mathbb{A}_s|\mathbb{A}_d] = \begin{bmatrix} 1 & \vdots & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \vdots & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & \vdots & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & \vdots & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \vdots & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & \vdots & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

With the help of the node-to-branch incidence matrix presented above, it can be observed that the voltage drop at each distribution line (*i.e.*, $\mathbb{E}_l$) can be defined as the difference of the voltage value at the ends of the line. These relations are listed in Equation (9).

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} = \begin{bmatrix} V_1\text{-}V_2 \\ V_2\text{-}V_3 \\ V_3\text{-}V_4 \\ V_2\text{-}V_5 \\ V_5\text{-}V_6 \\ V_2\text{-}V_7 \end{bmatrix}$$

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\mathbb{V}_s \text{-} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}\mathbb{V}_d$$

$$\mathbb{E}_b = \mathbf{A}_s\mathbb{V}_s + \mathbf{A}_d\mathbb{V}_d. \qquad (9)$$

Now, note that, if Kircchoff's first law is applied to each node in Figure 1 except the slack source, then the following relation is obtained:

$$\begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \\ J_6 \end{bmatrix}, \qquad (10)$$

which can be easily compacted with the help of the demandto-demand branch-to-node incidence matrix, $\mathbf{A}_d$, as follows:

$$\mathbb{I}_d = -\mathbf{A}_d^{\mathrm{T}}\mathbb{J}_b. \qquad (11)$$

Now, if Ohm's law, which was presented in Equation (5), is written using its inverse form (*i.e.*, $\mathbb{J}_b = \mathbb{Z}_{bb}^{-1}\mathbb{E}_b = \mathbb{Y}_{bb}\mathbb{E}_b$), then Equation (11) takes the following form:

$$\mathbb{I}_d = -\mathbf{A}_d^{\mathrm{T}}\mathbb{Y}_{bb}\mathbb{E}_b. \qquad (12)$$

In addition, if Equation (9) is replaced into (12), then the following result is obtained:

$$\mathbb{I}_d = -\mathbf{A}_d^{\mathrm{T}}\mathbb{Y}_{bb}\mathbf{A}_s\mathbb{V}_s\text{-}\mathbf{A}_d\mathbb{Y}_{bb}\mathbf{A}_d\mathbb{V}_d, \qquad (13)$$

which implies that this equation is pre-multiplied on both sides by $\left[\mathbf{A}_d^{\mathrm{T}}\mathbb{Y}_{bb}\mathbf{A}_d\right]^{-1}$. Furthermore, if the definition of the demanded current presented in (6) is considered, the following nonlinear power flow formula is reached:

$$\mathbb{V}_d = -\left[\mathbf{A}_d^{\mathrm{T}}\mathbf{Y}_{bb}\mathbf{A}_d\right]^{-1}\big(\mathbf{A}_d^{\mathrm{T}}\mathbf{Y}_{bb}\mathbf{A}_s\mathbb{V}_s + \mathrm{diag}^{-1}(\mathbb{V}_d^{\star})S_d^{\star}\big). \qquad (14)$$

Note that, if the iterative counter t is added to Equation (14), the BFPF with the recursive formula presented in (8) is reached, thus completing the demonstration.

## 4. Equivalence of The UTPF and the BFPF Approaches

To demonstrate that the UTPF formulation in (1) is completely equivalent to the BFPF approach in (8), let us consider the theorems and proofs presented below.

Lemma 3. *The upper-triangular matrix T and the demandto-demand branch-to-node incidence matrix are related with a negative inverse operation, i.e.,*

$$\mathbf{T} = -\left(\mathbf{A}_d^{\top}\right)^{-1}, \tag{15}$$

*which is fulfilled if and only if the distribution network has a strictly radial configuration.*

Proof. To demonstrate the aforementioned inverse relation between these matrices, it is necessary to substitute Equation (11) into (2), which yields

$$\mathbb{J}_b = -\mathbf{T}\mathbf{A}_d^{\top}\mathbb{J}_b, \tag{16}$$

thus implying that

$$-\mathbf{T}\mathbf{A}_d^{\top} = 1_{d\times d} \leftrightarrow \mathbf{T} = -\left(\mathbf{A}_d^{\top}\right)^{-1} \tag{17}$$

and completing the proof.

Lemma 4. *In the BFPF approach, the product between matrices regarding the voltage input meets the following criterion:*

$$-\left[\mathbf{A}_d^{\top}\mathbf{Y}_{bb}\mathbf{A}_d\right]^{-1}\mathbf{A}_d^{\top}\mathbf{Y}_{bb}\mathbf{A}_s = \mathbf{1}_d, \tag{18}$$

*if and only if the distribution network has a strictly radial topology.*

Proof. To demonstrate the relation presented in (15), it is necessary to consider the properties of the inverse of the product of the matrices. Thus, with three invertible square matrices (*i.e.*, **A**, **B**, and **C**) with dimensions $d\times d$,

$$\begin{aligned} \mathbf{C}&=\mathbf{AB} \\ \mathbf{C}^{-1}\,\mathbf{C}&=(\mathbf{AB})^{-1}\,\mathbf{AB} \\ \mathbf{1}_{d\times d}&=(\mathbf{B}^{-1}\,\mathbf{A}^{-1})\,\mathbf{AB}, \end{aligned}$$

which implies that $(\mathbf{AB})^{-1}= \mathbf{B}^{-1}\mathbf{A}^{-1}$. With this property, it can be noted that

$$\begin{aligned} -[\mathbf{A}_d^{\top}\mathbf{Y}_{bb}\mathbf{A}_d]^{-1}\mathbf{A}_d^{\top}\mathbf{Y}_{bb}\mathbf{A}_s &= 1_d, \\ -(\mathbf{Y}_{bb}\mathbf{A}_d)^{-1}(\mathbf{A}_d^{\top})^{-1}\mathbf{A}_d^{\top}\mathbf{Y}_{bb}\mathbf{A}_s &= 1_d, \\ -\mathbf{A}_d^{-1}\mathbf{Y}_{bb}^{-1}\mathbf{Y}_{bb}\mathbf{A}_s &= 1_d, \\ -\mathbf{A}_d^{-1}\mathbf{A}_s &= 1_d. \end{aligned} \tag{19}$$

In addition, if it is taken into account that $\mathrm{T}^{\top}$ is equal to $-\mathbf{A}_d^{-1}$, as demonstrated in (17), the following result is obtained:

$$\mathbf{T}^{\top}\mathbf{A}_s = 1_{\boldsymbol{d}}. \tag{20}$$

The result in (20) can be easily verified with the numerical example in Figure 1, as the $\mathbf{A}_s$ matrix only has a position with a value of one in the first row and the upper triangular matrix **T** is the sum of an identity matrix with an upper-triangular one, which clearly ensures that the product between both matrices produces a vector filled by ones, thus completing the demonstration.

Lemma 5. *The following matricial operations are equivalent for strictly radial distribution grids:*

$$\mathbf{T}^{\top}\mathbf{Z}_{bb}\mathbf{T} = \left[\mathbf{A}_d^{\top}\mathbf{Y}_{bb}\mathbf{A}_d\right]^{-1}. \tag{21}$$

Proof. To prove that both matricial operations are equivalent, the properties of the inverse matrices in (19) are considered, which yield the following result:

$$\mathbf{T}^{\top}\mathbf{Z}_{bb}\mathbf{T} = \mathbf{A}_d^{-1}\,\mathbf{Y}_{bb}^{-1}(\boldsymbol{A}_d^{\top})^{-1}\ , \tag{22}$$

where, if the definition in (15) is taken into account, the following result is reached:

$$\mathbf{T}^{\top}\mathbf{Z}_{bb}\mathbf{T} = \mathbf{T}^{\top}\,\mathbf{Y}_{bb}^{-1}\,\mathbf{T}^{\top}, \tag{23}$$

which confirms that the relation between the primitive impedance matrix and the primitive admittance matrix is an inverse operation, as defined in Equations (4) and (12), thus completing the proof.

Remark 1. *Note that, if the definitions in (18) and (21) are replaced into the BFPF recursive formula (8), it becomes the UTPF formula defined in (1). This is a theoretical confirmation that the UTPF and BFPF methods are mathematically equivalent.*

## 5. Convergence Analysis

This section addresses the convergence analysis of the UTPF method based on the demonstration presented by authors of [5] for the successive approximation power flow method. In this demonstration, the following assumptions are made [4], [13].

Assumption 1. *The operation of the voltage distribution network is far from the voltage collapse point, i.e., the amount of power consumption allows the solution of the active and reactive power balance constraints.*

Assumption 2. *The regulatory entities imposed a minimum positive regulation bound for the distribution grid's regular operation, i.e., $V^{\min} > 0$ exists.*

Assumption 3. The impedance matrix defined as $\mathbf{Z}_{bus} = \mathbf{T}^{\top}\mathbf{Z}_{bb}\mathbf{T}$ which relates the demand nodes among them is diagonally dominant, *i.e.*,$\left|\mathbf{Z}_{bus\,jj}\right| \geq \left|\mathbf{Z}_{bus\,jk}\right|, \forall j \neq k$ is always fulfilled.

To demonstrate the convergence properties of the UTPF method, we consider the formulation in (7), where an iterative counter is added, which produces:

$$\mathbb{V}_{\boldsymbol{d}}^{\boldsymbol{t+1}} = 1_d \mathbb{V}_s - \mathbf{Z}_{bus}\text{diag}^{-1}\left(\mathbb{V}_d^{t,\star}\right)S_d^{\star}, \quad (24)$$

where it is observed that it is a contraction map that can be defined using the Banach fixed-point theorem [13].

Theorem 6 (Banach fixed-point theorem). *The recursive UTPF formula presented by (24) is stable, and it is a contraction map that can be represented with a fixed-point structure as defined below*.

$$\mathbb{V}_{\boldsymbol{d}}^{\boldsymbol{t+1}} = \boldsymbol{G}(\mathbb{V}_d^t), \quad (25)$$

for an initial point $\mathbb{V}_d^0$ such that $\mathbb{V}_d$ fulfills Assumption 1, and

$$\|\boldsymbol{G}(\mathbb{V}_d^0) - \boldsymbol{G}(\mathbb{U}_d)\| \leq \gamma\|(\mathbb{V}_d^0) - \mathbb{U}_d\|, \quad (26)$$

being $\mathbb{U}_d$, the vector of voltages that solves the iterative formula (24), and γ is a constant parameter defined in the real domain that is contained in the interval [0, 1].

Proof. The iterative power flow formula (23) that represents the UTPF method can be rewritten as follows:

$$\mathbb{V}_{\boldsymbol{d}}^{\boldsymbol{t+1}} = \boldsymbol{G}(\mathbb{V}_d^t) = 1_d \mathbb{V}_s - \mathbf{Z}_{bus}\left[\frac{S_{dk}^{\star}}{\mathbb{V}_{dk}^{t,\star}}\right]_{k\epsilon\mathcal{D}} \quad (27)$$

where $\mathcal{D}$ is the set containing all the demanded nodes, i.e., all the nodes except the substation bus.

Now, considering the general form of the recursive formula of the UTPF method and its relation with the fixedpoint definition in (24), it is observed that the vector of voltage solutions $\mathbb{U}_d$ complies with $\mathbb{U}_d$ = **G** ($\mathbb{U}_d$). In addition, it is a unique solution of the power flow problem if and only if there is a contraction map $g$ ($\mathbb{U}_d$) on $\mathbb{V}_d$, this is:

$$\begin{aligned}\left\|\mathbb{V}_{\boldsymbol{d}}^{\boldsymbol{t+1}} - \mathbb{U}_d\right\| &= \left\|\mathbf{G}\left(\mathbb{V}_{\boldsymbol{d}}^{\boldsymbol{t+1}}\right) - \mathbf{G}(\mathbb{U}_d)\right\| \\ &\left\|\mathbf{Z}_{bus}S_d^{\star}\left[\frac{1}{\mathbb{U}_{dk}^{\star}} - \frac{1}{\mathbb{V}_{dk}^{t,\star}}\right]_{k\epsilon\mathcal{D}}^{\top}\right\| \\ &\left\|\mathbf{Z}_{bus}S_d^{\star}\left[\frac{\mathbb{V}_{dk}^{t,\star}}{\mathbb{U}_{dk}^{\star}} - \frac{\mathbb{U}_{dk}^{\star}}{\mathbb{V}_{dk}^{t,\star}}\right]_{i\epsilon\mathcal{D}}^{\top}\right\| \\ &\leq \gamma\|\mathbb{V}_d^t - \mathbb{U}_d\|,\end{aligned} \quad (28)$$

being the γ parameter defined as follows:

$$\gamma = \frac{\|\mathbf{Z}_{bus}S_d^{\star}\|}{(V^{min})^2} \quad (29)$$

In this point, if we consider Assumption 2 which defined the nature of the matrix $\mathbf{Z}_{bus_{kk}}$, then Equation (29) can be reformulated as:

$$\gamma = \max_{k\epsilon\mathcal{D}}\left\{\frac{\left|\mathbf{Z}_{bus_{kk}}\right|\left|S_{dk}^{\star}\right|}{(V^{min})^2}\right\} \quad (30)$$

Observed that considering that $\mathbf{Z}_{bus_{kk}}$ is the Thévenin impedance at node k [13], and based on the mathematical form exhibited by γ in (30), the following equivalence is reached.

$$\gamma = \max_{k\epsilon\mathcal{D}}\left\{\frac{\frac{|S_{di}^{\star}|}{V^{min}}}{\frac{V^{min}}{\left|Z_{bus_{kk}}\right|}}\right\}, \quad (31)$$

which clearly shows that the parameter γ is contained in the interval [0, 1] since:

i. The denominator of (31) corresponds to the minimum short-circuit current at node k, and
ii. The numerator corresponds to the maximum demanded current possible at node k, which in normal operating conditions is always lower that the short-circuit current.

The aforementioned operative conditions demonstrate that the UTPF formula (23) allows solving the power flow problem ensuring the stability and convergence of the iterative process, which completes the proof [4], [5].

## 6. Numerical Validations

To validate the theoretical achievement regarding the equivalence between the UTPF and BFPF methods numerically, simulations in well-known single- and three-phase networks are presented. The IEEE 33-, 34-, 69-, and 85bus grids are considered for validating the equivalence between both power flow methods in single-phase distribution grids. The complete parametric information and the grid configurations for each one of these distribution networks can be consulted in [7] and [5]. Table 1 presents the comparative results for each test feeder.

Table 1. Numerical performance of the BFPF and the UTPF methods in single-phase networks

| Method | Power losses (kW) | Iterations | Proc. time (ms) |
|---|---|---|---|
| **IEEE 33-bus system** | | | |
| BFPF | | 10 | 0.5777 |
| UTPF | 210.97850 | 10 | 0.0797 |
| **IEEE 34-bus system** | | | |
| BFPF | | 8 | 0.6405 |
| UTPF | 221.75236 | 8 | 0.1662 |
| **IEEE 69-bus system** | | | |
| BFPF | | 10 | 2.3523 |
| UTPF | | 10 | 0.1856 |
| **IEEE 85-bus system** | | | |
| BFPF | | 11 | 2.7924 |
| UTPF | 225.07147 | 11 | 0.5450 |

Numerical results in Table 1 show that as theoretically expected, the TPB and BFPF approach solve the power flow problem with a convergence error $\varepsilon = 1 \times 10^{-10}$ with the equal number of iterations, finding the identical value regarding the total grid power losses. Nevertheless, as previously demonstrated in the specialized literature (see [7] and [14]), the TBPF method solves the power flow problem for radial distribution networks faster than the BFPF approach. Note that it is possible since the upper triangular incidence matrix is non-built with the inverse of the incidence matrix; it is constructively formed following the algorithm reported by authors of [10], which helps with the reduction of processing times by avoiding inverting matrices.

To confirm the equivalence between the TBPF and the BFPF approaches to three-phase networks, we select three IEEE test feeders composed of 8, 25, and 37 nodes reported in [15] for the optimal-phase swapping problem. Table 2 lists the comparative results between both power flow approaches for three-phase asymmetric networks.

Results in Table 2 confirmed that both power flow approaches require the same number of iterations to fulfill the expected convergence error; also, for three-phase grids, the expected processing time is better for the TBPF method when compared with the BFPF approach. Finally, Figure 2 presents the relation between the processing times of both power flow methods, which permits observing that in all the simulations (single- and three phase networks), the TBPF is faster than the BFPF approach.

Table 2. Numerical performance of the BFPF and the UTPF methods in three-phase networks

| Method | Power losses (kW) | Iterations | Proc. time (ms) |
|---|---|---|---|
| **IEEE 8-bus system** | | | |
| BFPF | | 5 | 0.8224 |
| UTPF | 13.9925 | 5 | 0.4358 |
| **IEEE 25-bus system** | | | |
| BFPF | | 9 | 3.2530 |
| UTPF | 75.4207 | 9 | 1.6923 |
| **IEEE 37-bus system** | | | |
| BFPF | | 9 | 6.2549 |
| UTPF | 76.1357 | 9 | 3.9450 |

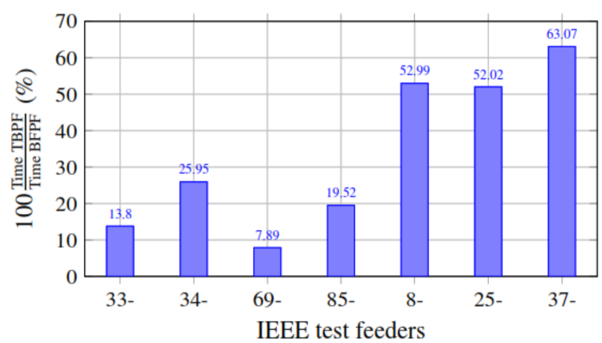


Figure 2. Performance behavior of the processing times in the single- and three-phase networks

Figure 2 shows that in the case of the single-phase networks, the requested processing times are lower, which is expected the required calculations are lower when compared with three-phase networks, where additional calculations are required previous to the usage of the power flow formula at each iteration. These calculations are related to the demand currents that, in the three-phase case, can be produced by triangle- or star-connected loads.

## 7. Conclusion

This paper presented a theoretical demonstration of the equivalence between the upper-triangular power flow method and the backward/forward approach, which uses its matricial equivalent via branch-to-node incidence matrices. The mathematical procedures confirmed that both recursive power flow formulas are completely equivalent if the distribution network under analysis has a radial topology with a single slack node. Note that the only assumption for all the proofs presented in this paper is that the distribution network is ordered, *i.e.*, the slack bus is connected at node 1, and the remaining nodes (i.e., *2*, ..., *n*) are associated with demand and step nodes. Numerical results in the IEEE single-phase networks composed of 33, 34, 69, and 85, and the IEEE three-phase grids composed of 8, 25, and 37 nodes, confirmed, as theoretically prognosticated, that to reach the same desired convergence, the TBPF, and the BFPF approach would take the equal number of iterations. In addition, numerical results in those systems confirmed that the TBPF approach is faster than the BFPF method for solving the power flow problem in radial distribution grids.

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The authors reported no potential conflict of interest.

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