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A brief solution to three-body problem: Newtonian and Hamiltonian versions

Una breve solución al problema de los tres cuerpos: Versiones Newtoniana y Hamiltoniana

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Abstract

The problem of the three bodies was cataloged as one of the best-positioned problems and the pinnacle of functional analysis by Poincaré himself when he discovered that the problem itself presents a chaotic behavior and that it was impossible to apply integrable methods to this system. Therefore, its analytical solution was impossible to obtain, since its solution strongly depended on the initial conditions (weak chaos). With the development of modern numerical methods, together with the immense advances in the hardware of the new computers, attempts have been made to attack this system from different schemes and numerical stencils, to describe the main physical properties of this system (the trajectory is only one of these). With this, in the present work, we will study this problem from the Newtonian and Hamiltonian versions and the restricted problem. Special interest will be devoted to the numerical analysis of this system, The work focuses on a pedagogical description of the topic (constructivist), academic clarity, and application of numerical analysis.

Keywords: Toroidal geometry; Maxwell's equations; Numerical methods; Hamiltonian; Lagrangian.

Resumen

En esta contribución, estudiamos las oscilaciones del potencial eléctrico en una película delgada superconductora mesoscópica cuando se aplica una corriente externa. Analizamos la resistividad y el potencial eléctrico en función de la corriente aplicada para varios campos magnéticos externos aplicados y el tamaño de la muestra. Además, hemos calculado el potencial eléctrico en función del tiempo característico. Para estudiar este problema, resolvemos las conocidas ecuaciones generalizadas de Ginzburg-Landau dependientes del tiempo utilizando el método de variable de enlace. Encontramos que la corriente crítica disminuye cuando aumenta el campo magnético externo y disminuye el tamaño de la muestra. Además, la frecuencia de oscilación de los vórtices cinemáticos, evidenciada en las oscilaciones del potencial eléctrico, es altamente dependiente del campo magnético aplicado.

Palabras clave: Ginzburg-Landau, Corriente crítica, Superconductividad, Estado de vórtices, Mesoscópico.

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1. Introduction

In the field of space debris tracking and management, understanding the motion of debris objects influenced by the gravitational forces of Earth, the Moon, and the Sun is critical. The three-body problem can be used to predict the future positions of space debris and plan collision avoidance maneuvers for operational satellites. While this problem is challenging and analytically unsolvable in most cases, numerical methods and computer simulations are commonly used in engineering applications to approximate solutions and make practical predictions. These applications demonstrate how celestial mechanics concepts, including the three-body problem, are essential for the successful planning and execution of various engineering endeavors in the realm of space exploration and satellite technology. With this, it is extremely important that undergraduate students become familiar with this type of modern problems. In this way, the problem of the three bodies, from the Newtonian vision of the particles, is defined by three bodies with different initial positions and velocities, subjected to gravitational interaction between them depending on the position, which comply with Newton's third law in its weak and strong parts [1], [2] that together with Newton's second law, considering a system with constant mass and in an inertial reference frame, can be presented as a differential system of equations $\mathbf{f} = \dot{\mathbf{P}} = \mathrm{m}\ddot{\mathbf{r}} = -\mathrm{G}\sum_{\substack{i=1 \\ i\neq j}}^{n} m_i m_j \left|\boldsymbol{r}_j - \boldsymbol{r}_i\right|^{-3} (\boldsymbol{r}_j - \boldsymbol{r}_i)$ [3]. The $\boldsymbol{r_j}$ are defined as the positions of the punctual particles in the inertial reference frame and m_j, the masses of each of the bodies, and G the gravitational constant. The first approach to the study of this system was given by Jacobi [4], [5] who starts from the fact already known for the two-body system; in which, the study of the system is simpler, with respect to the center of mass $\boldsymbol{R}$, which is defined by the positions of each of the particles in the inertial system and the vectors from the center of mass to each of the particles $\boldsymbol{r}$, $\mathbf{R} = \sum_k m_k \boldsymbol{r}_k (\sum_k m_k)^{-1}$; $\boldsymbol{r} = \boldsymbol{r}_k - \boldsymbol{R}, k = 1,2$.

The general idea is to initially study the two-body system, also called a binary system, and then study the position of the third body with respect to this binary center of mass. Thus, the position of the particles to the binary center of mass would have the following form [3], [4], [5] (see Figure 1, where we show the position vectors with respect to the inertial frame and with respect to the center of mass). We note that $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = 0$ describes the lineal momentum conservation $\boldsymbol{p}$ with respect to this point [2]. Now, defining the position of the third body with respect to said center of mass and between the particles, in addition, we obtain equation (1):

$$\boldsymbol{r}_3 - \boldsymbol{r}_1 = \boldsymbol{R} + \boldsymbol{r}_3 - (\boldsymbol{R} + \boldsymbol{r}_1) = \boldsymbol{r}_3 + \frac{m_2(r_2 - r_1)}{m_1 + m_2};$$
$$\boldsymbol{r}_3 - \boldsymbol{r}_2 = \boldsymbol{R} + \boldsymbol{r}_3 - (\boldsymbol{R} + \boldsymbol{r}_2) = \boldsymbol{r}_3 + \frac{m_1(r_2 - r_1)}{m_1 + m_2}; \quad (1)$$

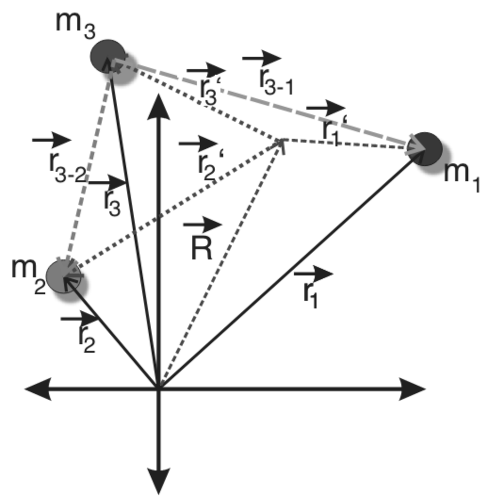

Figure 1. Position vectors for the three-body system with respect to an inertial frame and relative coordinates for each of the particles with respect to $\boldsymbol{R}$. Observe that $\boldsymbol{r_3}$ is taken with respect to the center of mass, of the binary system (equation (2)).

Additionally, by applying the following property (equation (2)), defined for the relative position of the third particle:

$$\boldsymbol{r}_3 = \frac{m_1}{m_1 + m_2}(\boldsymbol{r}_3 - \boldsymbol{r}_1) + \frac{m_2}{m_1 + m_2}(\boldsymbol{r}_3 - \boldsymbol{r}_2) \quad (2)$$

and with the help of equation (1) we can express the acceleration, for the binary system described by equation (3):

$$\ddot{\boldsymbol{r}}_1 = -G\left|\frac{m_2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}(\boldsymbol{r}_2 - \boldsymbol{r}_1) + \frac{m_3}{|\boldsymbol{r}_3 - \boldsymbol{r}_1|^3}(\boldsymbol{r}_3 - \boldsymbol{r}_1)\right|$$
$$\ddot{\boldsymbol{r}}_2 = -G\left|\frac{m_3}{|\boldsymbol{r}_3 - \boldsymbol{r}_2|^3}(\boldsymbol{r}_3 - \boldsymbol{r}_2) + \frac{m_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right| \quad (3)$$

After subtracting the two expressions and deriving expression (6), the equations of motion for the system are obtained. This is the first approximation, for the solution of the system of the three bodies. Already as initially perceived by Poincaré [1], [2], [3], [4]. The problem drastically depends on the initial conditions of all the particles and in general the prediction of the orbits, it is a problem with weak chaos and the solutions of said system can present strange attractors in the evolution of the nonlinear system [2]. However, the technique used to solve this problem, with good theoretical results, is based on the numerical analysis of the system [6]. In the present work, we will show the main results of the study of the three-body system, from the visions of Newtonian and Hamiltonian mechanics.

2. Integration of the orbits

One of the most complex problems in the description of the orbits for the system of the three bodies, lies in the difficulty of the realization of the integration (quadrature), since the interaction between them at certain points changes direction and magnitude. very quickly, limiting the temporal variation when performing the integrations, for each of the times. Thus, an initial version for this integration is to perform an approximation via the Taylor series [6], [7], for the position at each time, understanding that the orbital function must be well-behaved and meet some minimum requirements from the functional analysis [8]. Thus, for the first particle:

$$\boldsymbol{r}_1(t+\Delta t) = \boldsymbol{r}_1 + \dot{\boldsymbol{r}}_1\Delta t + \frac{(\Delta t)^2}{2!}\ddot{\boldsymbol{r}}_1 + \frac{(\Delta t)^3}{3!}\dddot{\boldsymbol{r}}_1 + \cdots \quad (4)$$

Due that $\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1$, y $\dddot{\boldsymbol{r}}_1 = \dot{\boldsymbol{F}}_1$ in equation (4), we will proceed to derive equation (1) with respect to time, and equating in equation (4) the temporal evolution for each of the position vectors is obtained:

The complete solution of the orbits is obtained by extending this expression for each of the particles. Thus, for the numerical solution of this expression, we start from the initial positions, velocities, and masses of each one of the particles, together with an adequate passage of time [9], [10], [11], since the approximation can be unstable in the strict sense, i.e the accumulation of errors for each of the steps when integrating [12], [13], [14], Thus, in Figure 2, we present the orbits, for a single body (a), two bodies (b) and finally for three bodies (c), for given masses, as well as the initial position and velocity conditions for each body. Integration, from this Newtonian perspective, is complex and requires a dense mesh for the time step.

$$\begin{aligned}
&\boldsymbol{r}_1(t+\Delta t) \\
&= \boldsymbol{r}_1 + \dot{\boldsymbol{r}}_1\Delta t \\
&- \mathrm{G}\sum_{\substack{i=1\\ i\neq j}}^{3} m_i \left| \frac{(\Delta t)^2}{2!}\frac{(\boldsymbol{r}_j-\boldsymbol{r}_i)}{|\boldsymbol{r}_j-\boldsymbol{r}_i|^3} \right. \\
&+ \frac{(\Delta t)^3}{3!}\frac{(\dot{\boldsymbol{r}}_j-\dot{\boldsymbol{r}}_i)}{|\boldsymbol{r}_j-\boldsymbol{r}_i|^3} \\
&- 3\frac{(\boldsymbol{r}_j-\boldsymbol{r}_i)[(\boldsymbol{r}_j-\boldsymbol{r}_i)\cdot(\dot{\boldsymbol{r}}_j-\dot{\boldsymbol{r}}_i)]}{|\boldsymbol{r}_j-\boldsymbol{r}_i|^5} \\
&+ \frac{(\Delta t)^4}{4!}\frac{(\ddot{\boldsymbol{r}}_j-\ddot{\boldsymbol{r}}_i)}{|\boldsymbol{r}_j-\boldsymbol{r}_i|^3} \\
&- 6\frac{(\dot{\boldsymbol{r}}_j-\dot{\boldsymbol{r}}_i)[(\boldsymbol{r}_j-\boldsymbol{r}_i)\cdot(\dot{\boldsymbol{r}}_j-\dot{\boldsymbol{r}}_i)]}{|\boldsymbol{r}_j-\boldsymbol{r}_i|^5} \\
&+ 3\frac{(\boldsymbol{r}_j-\boldsymbol{r}_i)}{|\boldsymbol{r}_j-\boldsymbol{r}_i|^5}\left\{5\frac{(\boldsymbol{r}_j-\boldsymbol{r}_i)\cdot(\dot{\boldsymbol{r}}_j-\dot{\boldsymbol{r}}_i)}{|\boldsymbol{r}_j-\boldsymbol{r}_i|^2}\right. \\
&- (\dot{\boldsymbol{r}}_j-\dot{\boldsymbol{r}}_i)\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i) - (\boldsymbol{r}_j-\boldsymbol{r}_i) \\
&\left.\left.\cdot(\ddot{\boldsymbol{r}}_j-\ddot{\boldsymbol{r}}_i)\right\} + \cdots \right|
\end{aligned} \quad (5)$$

The second vision for the study of the variables of the system of the three bodies, rests on the Lagrangian-Hamiltonian vision of the system, through the definition of the Lagrangian scalar functional and its respective Legrende (Hamiltonian) transformation. Because the interaction between particles does not depend on velocity, the Hamiltonian function represents energy and is generally simpler to solve than its Newtonian counterpart [12]. In the Lagrangian-Hamiltonian description, the vision of analytical mechanics, starts from the definition of a Lagrangian scalar function, dependent on generalized positions and velocities, in addition to the temporal parameter and its respective generalized moments [1], [4], [6], $p(q,\dot{q},t) = \frac{\partial \mathcal{L}}{\partial \dot{q}}$, by deriving the Lagrangian variationally and with the help of the fundamental lemma of calculus [5], [6], the Euler-Lagrange equation is obtained (equation (6)):

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{\partial \mathcal{L}}{\partial t} = 0 \quad (6)$$

In addition, through the Legendre transform of the Lagrangian function, the Hamiltonian of the mechanical system [3], [5] is obtained:

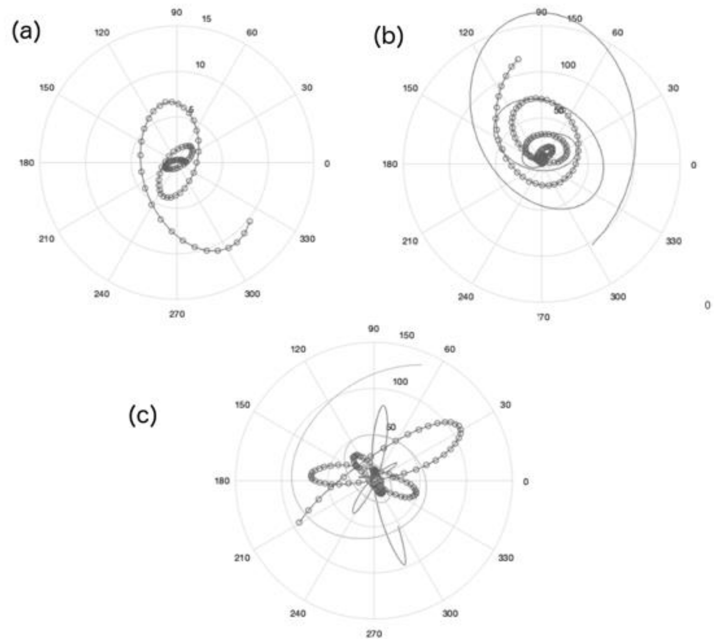

Figure 2. Orbit for (a) a body, subjected to the gravitational force of another fixed located in $\boldsymbol{r} = 0$ (b) Orbits for two bodies, with gravitational interaction dependent on $\boldsymbol{r_2} - \boldsymbol{r_1}$ and (c) Orbit for three bodies with mutual interaction, described by equation (8). The masses used are: $m_1 = 1.0, m_2 = 0.1$, y $m_3 = 0.5$. $r_{01} = 0.0, v_{01} = 0.0, r_{02} = 3, v_{02} = 2, r_{03} = 0.5, v_{03} = 3$; $m_1 = 1, m_2 = 1, m_3 = 2, G = 4\pi^2$; $\varepsilon = 10^{-7}$; $N_T = 200$.

$$\mathcal{H} = \sum_{a=1}^{n} \dot{q}_a p_a - \mathcal{L}; \dot{q} = \frac{\partial \mathcal{H}}{\partial p}; \ \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \qquad (7)$$

Being a the degrees of freedom of the system (equation (7)). With this, let us suppose a functional A, dependent on the generalized positions and moments $A(q,p)$, and taking its total derivative, we obtain equation (11):

$$\frac{dA}{dt} = \frac{\partial A}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial t} = \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p} \qquad (8)$$

Equation (8) describes the physical fact that describes whether the conjugate variables are independent or not.

3. Restricted three-body problem

For the study of the three bodies in a restricted form, several considerations are made that simplify the problem. The first is that the mass of the third body is very small compared to the initial two. The following is to study a new system, with origin in the center of mass (Sidereal system), and then define a new system that rotates an angle of θ with respect to this inertial system (Sinodic system). Additionally, the total mass is considered as a unit $(m_1 + m_2 = 1.0)$ and the distance between the two primary masses (bodies with greater mass), also as the unit. Having defined this, in Figure 3 we present the three frames of reference for the study of the restricted system (ξ, η) (Sideral), (ξ^*, η^*) (Synodic) and the initial one, which we simply denote as inertial. We will proceed to find the Hamiltonian in the sidereal coordinates (equation (8)), taking into account that we do not consider potentials dependent on speed or time.

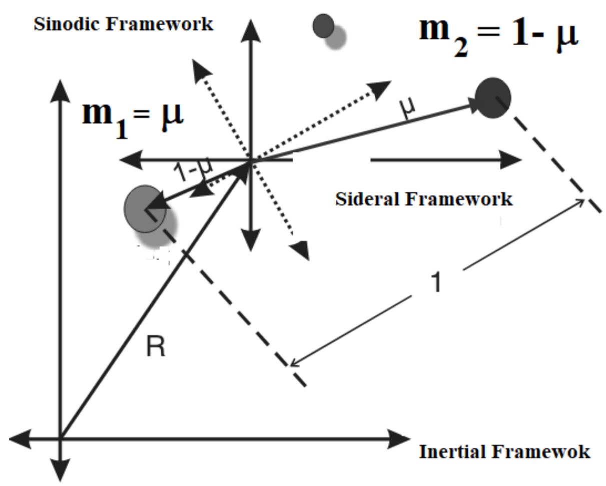

Figure 3. Reference frames used for the study of the restricted three-body problem.

Thus, we obtain:

$$H = \frac{1}{2}\left|P_\xi^2 + P_\eta^2\right| - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} \quad (9)$$

where $\rho_1 = \sqrt{(\xi + \mu cos\theta)^2 + (\eta + \mu sin\theta)^2}$ and $\rho_2 = \sqrt{(\xi - (1-\mu)cos\theta)^2 + (\eta - (\eta - (1-\mu)sin\theta)^2}$, which would describe the system in the Sidereal reference frame. Now, to perform the transformation of (ξ, η) to (ξ^*, η^*), which is the transformation of a rotated system to a stationary one, we do it by using the matrix of director cosines [3] $\xi^* = \xi cos\theta - \eta sin\theta$ y $\eta^* = \xi sin\theta + \eta cos\theta$, with which, we can define the canonical transformation between the Hamiltonians, by means of a generating function F that depends on time and is transformed, to obtain this new Hamiltonian [1], [2], [3]: $H^* = H + \frac{\partial F}{\partial t}$, con $F = -(\xi cos\theta - \eta sin\theta)p_\xi - (\xi sin\theta + \eta cos\theta)p_\eta$ and deriving the generating function, we get the equation (10):

$$H = \frac{1}{2}\left|P_\xi^2 + P_\eta^2\right| - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} + \xi cos\theta p_\eta + \eta sin\theta p_\eta + \xi sin\theta p_\xi + \eta cos\theta p_\xi \quad (10)$$

With the application of equations (10), we obtain the equations of motion, in Synodic coordinates (equation (15) to equation (18)):

$$H = \frac{1}{2}\left|P_\xi^2 + P_\eta^2\right| - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} + \xi cos\theta p_\eta + \eta sin\theta p_\eta + \xi sin\theta p_\xi + \eta cos\theta p_\xi \quad (10)$$

$$\dot{\xi} = p_\xi + \eta \quad (11)$$

$$\dot{\eta} = p_\eta - \xi \quad (12)$$

$$\dot{p_\xi} = p_\eta + (1-\mu)(\xi + \eta)((\xi + \eta)^2 + \eta^2)^{-3/2} \quad (13)$$

$$\dot{p_\eta} = -p_\xi + (\mu\eta)((\xi + \eta)^2 + \eta^2)^{-3/2} \quad (14)$$

Finally:

$$\ddot{\xi} = 2\dot{\eta} + \xi + (1-\mu)(\xi + \eta)((\xi + \eta)^2 + \eta^2)^{-3/2} - (\mu(\xi - (1-\mu)))((\xi - (1-\mu) + \eta^2)^{-3/2} \quad (15)$$

$$\ddot{\eta} = -2\dot{\xi} + \eta + (1-\mu\eta)((\xi + \eta)^2 + \eta^2)^{-3/2} - \mu\eta((\xi - (1-\mu) + \eta^2)^{-3/2} \quad (16)$$

That is the system of equations that must be solved, for the accelerations in the Synodic reference frame. Differential expressions must be solved in a self-consistent manner. However, there is a simpler way to approach them, the one initially given by Jacobi, in which it is considered $\dot{\xi} = \dot{\eta} = 0$, which in turn allows defining $\ddot{\xi} = \ddot{\eta} = 0$, then, by equating both expressions present in equation (11) and in equation (16), to obtain the combination of possible values of μ y ξ, for a determined value of η. thus:

$$\ddot{\eta} = -2\dot{\xi} + \eta + (1-\mu\eta)((\xi + \eta)^2 + \eta^2)^{-3/2} - \mu\eta((\xi - (1-\mu) + \eta^2)^{-3/2} \quad (17)$$

Thus, there exists a trivial solution for the variable η what happens for $\eta = 0$ in the equation (21). With which, considering the functional form, we must find the roots of the expression:

$$\xi + (1-\mu)(\xi + \eta)((\xi + \mu))^{-3} - \Big(\mu(\xi - (1-\mu))\Big)((\xi - (1-\mu))^2)^{-3/2} = 0 \Rightarrow \xi + (1-\mu)(\xi + \mu)^2 - \mu\big(\xi - (1-\mu)\big)^2 = 0 \quad (18)$$

To obtain the roots of equation (22), different numerical methods [15], [16] can be used. One of the simplest is bisection, which is based on the idea that the product of the function evaluated at two points, must be negative $f(a)f(b) < 0$, $f(\xi, \mu)$ with μ as an initial value [15]. For this method, a teach of the iterations, it bisects the domain and divides by finding a midpoint. Thus, suppose that x_1 and x_2, are the two bounds, where the root is assumed to exist, the function is evaluated at both points, and the function is evaluated $f(x_1)$ and $f(x_2)$, with $c = (x_1 + x_2)/2$ and $f(c)$. Given this, the products of the functions are realized, $f(x_1)f(c)$ y $f(x_2)f(c)$ are evaluated and, for the negative value that is in said interval, it is transformed $x_1 = c$, o $x_2 = c$, (according to whether the second or the first product is negative), and the process is repeated for each iteration. Because of this, the interval is divided at each iteration of the form $\epsilon = \frac{dx}{2^n}$, where $dx = x_2 - x_1$, and ϵ, the tolerance, for obtaining the root, which describes the difference between two values obtained in two iterations for obtaining the root, which in the present case we use as $\epsilon = 10^{-7}$. With this, we present in Figure 4 the roots obtained for the variable ξ for different values of a fixed initial variable μ. With that, each point in the Synodic system must be given by the set of values (ξ, μ), in this particular case with η, which was the value taken for the solution of equation (18). Following this, the process can be repeated for another value of η to obtain the new coordinates of (ξ, η).

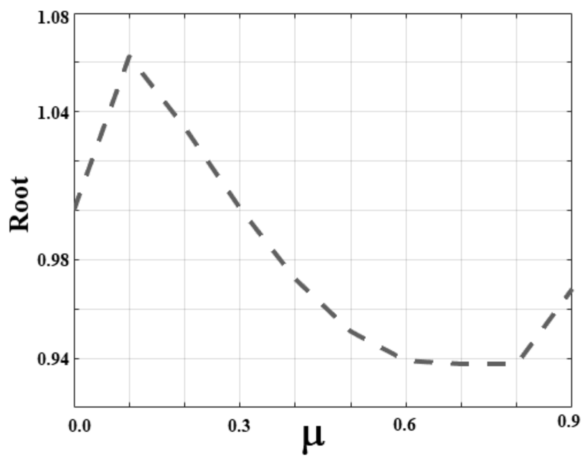

Figure 4. Roots for ξ as a function of different values of μ. $\epsilon = 10^{-6}$.

4. Conclusions

In the present work, we approached the problem of the three bodies from two visions, Newtonian mechanics and Hamiltonian dynamics. The general idea was to start from the main concepts and work towards addressing the problem in general terms. Given this, all the conceptual bases in both mechanical visions have been developed and the general ways of approaching the complex problem have been explained. However, the equations of the orbits were solved numerically, for one, two, and three bodies for certain specific parameters of mass and position and subjected to gravitational forces dependent on the relative positions. Then, we approached from the Hamiltonian vision, the restricted problem, and through the development of the canonical transforms the main expression of the problem was obtained. Then, through a simplification, the expression for the nodes in the Synodic reference frame was obtained, to which a numerical method (bisection) was applied to find the roots of the expression ξ for certain values of μ that, together with η, determine the positions of the three bodies.

Finally, we highlight the fact that the work has been built without omitting any step in the mathematical development, applying clarity in the physical concepts and making a structured numerical implementation to two current problems, with results that show novelty in the numerical treatment and with excellent coincidence with different recent papers.

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Autor Contributions

C. Aguirre -Tellez: built the computer code, analyzed the numerical elements to analyze the sample, analyzed the data and wrote the paper. M. Rincón-Joya: analyzed the results obtained and wrote the paper. J. J. Barba-Ortega: built the computer code, analyzed the numerical elements to analyze the sample, analyzed the data and wrote the paper.

Conflicts of Interest

The authors declare that there is no conflict of interests of any kind regarding the publication of the results of our research work.

Institutional Review Board Statement

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Informed Consent Statement

Not applicable.

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